Math 255A Lecture 21 Notes

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1 Spectral Theory for Compact Operators

1.1 Applications of analytic Fredholm theory

Last time, we proved analytic Fredholm theory, which said that if $T(z)$ is a family of operators in $\mathcal{L}(B_1, B_2)$ that is holomorphic in z (in some domain $\Omega \subseteq \mathbb{C}$) and if $T^{-1}(z_0)$ exists for some $z_0 \in \Omega$, then $\Sigma = \{z \in \Omega : T(z) \text{ is not invertible}\}$ is a discrete subset of Ω.

Example 1.1. Let M be a compact C^∞ Riemannian manifold (e.g. torus, sphere,etc.). Let $V \in L^\infty(M, \mathbb{C})$, and consider the **Schrödinger operator** $P = -\Delta + V : H^2(M) \to L^2(M)$. where $H^2(M) = \{u \in L^2(M) : \partial^\alpha u \in M^2(M), |\alpha| \leq 2\}$ is a Sobolev space. What is $\mathrm{Spec}(P)$? We need 2 basic facts (that we will accept without proof).

Proposition 1.1. *The inclusion map $H^2(M) \to L^2(M)$ is compact.*

Proposition 1.2. *For all $x \in \mathbb{C} \setminus \mathbb{R}$, $-\Delta - zI : H^2 \to L^2$ is bijective and*

$$\|(-\Delta - zI)^{-1}\|_{\mathcal{L}(L^2,L^2)} \leq \frac{1}{\mathrm{Im}(z)}.$$

This second fact follows form the fact that $-\Delta$ is self adjoint. Now observe that

$$P - z = \underbrace{(-\Delta + iI)}_{\text{bijective}} + \underbrace{B - zI - iI}_{\text{compact}},$$

so $P - z$ is Fredholm of index 0 and is holomorphic in z. We claim that there exists some $z_0 = it$ such that $P - z_0 I : H^2 \to L^2$ is bijective. Write

$$P - z_0 I = -\Delta + V - z_0 I = (I + V(-\Delta - z_0)^{-1})(-\Delta - z_0).$$

To show that $(I + V(-\Delta - z_0)^{-1})$ is invertible, we can make $\|V(-\Delta - z_0)^{-1})\| < 1$. So

$$\|V(-\Delta - z_0)^{-1}\| \leq \frac{\|V\|_{L^\infty}}{|\,\mathrm{Im}(z_0)|},$$

so we can take z_0 with large enough imaginary part to make this small. By the analytic Fredholm theory, we get $\mathrm{Spec}(P) \subseteq \mathbb{C}$ is discrete and $\mathrm{Spec}(P) \subseteq \{z : |\,\mathrm{Im}(z)| \leq C\}$. Moreover, the spectrum consists entirely of eigenvalues.

Example 1.2. When operators are not Fredholm, the spectrum may not have eigenvalues. Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be $u(x) \mapsto \sin(x)u(x)$. Then $T \in \mathcal{L}(P^2, L^2)$, and $\operatorname{Spec}(T) = [-1, 1]$, while T has no eigenvalues. If $Tu = \lambda u$, for $\lambda\ in \mathbb{C}$, then $u = 0$ a.e. Take $\lambda \in [-1, 1]$, and show that an estimate of the form $\|u\|_{L^2} \leq C\|(T - \lambda I)u\|_{L^2}$ cannot hold: $m(\underbrace{\{x : |\sin(x) - \lambda| < \varepsilon\}}_{E_{\lambda,\varepsilon}}) > 0$ for all $\varepsilon > 0$. Letting $u = \mathbb{1}_{E_{\lambda,\varepsilon}}/m(E_{\lambda,\varepsilon})^{1/2}$, sending $\varepsilon \to 0$ gives us the conclusion.

1.2 Spectral theory for compact operators

Theorem 1.1 (spectral theory for compact operators)**.** *Let B be an infinite dimensional Banach space, and let $T : B \to B$ be compact.*

1. $0 \in \operatorname{Spec}(T)$.
2. *If* $0 \neq \lambda \in \operatorname{Spec}(T)$, *then* $\ker(T - \lambda I) \neq 0$.
3. *One of the following occurs:*
   (a) $\operatorname{Spec}(T) = \{0\}$.
   (b) $\operatorname{Spec}(T) \setminus \{0\}$ *is a finite set.*
   (c) $\operatorname{Spec}(T) \setminus \{0\}$ *is a countable set* $= \{\lambda_1, \lambda_2, \dots\}$, *and* $\lambda_n \to 0$.

Proof. These statements are consequences of the results we have already proven.

1. This follows from Riesz's theorem.
2. If $\lambda \neq 0$¡ then $T - \lambda I = (-\lambda)(I - (1/\lambda)T)$ is Fredholm of index 0. If $\lambda \in \operatorname{Spec}(T)$, then $\ker(T - \lambda I) \neq \{0\}$.
3. Apply the analytic Fredholm theory to $F(\lambda) = (-\lambda)(I - (1/\lambda)T)$. $F(\lambda)$ is invertible for large λ. So $\operatorname{Spec}(T) \setminus \{0\}$ consists of at most countably many isolated points. □

Example 1.3. Let $B = L^2(0, 1)$, and let $Tf(x) = \int_0^x f(y)\,dy$ be the **Volterra operator**. T is compact. We claim that $\operatorname{Spec}(T) = \{0\}$. If $\lambda \in \operatorname{Spec}(T) \setminus \{0\}$, then there exists some $f \in L^2$ such that $\int_0^x f(y)\,dy = \lambda f(x)$. This implies $f(x) = \lambda f'(x)$ with $f(0) = 0$. So $f = 0$.

Let $\lambda \in \operatorname{Spec}(T) \setminus \{0\}$, and consider the resolvent of T in a neighborhood of λ. We have the Laurent expansion $(T - (\lambda + z)I)^{-1} = \sum_{j=-N}^{\infty} A_j z^j$ for $0 < |z|$ small, $1 \leq N < \infty$, and $A_j \in \mathcal{L}(B, B)$, where $A_{-N}, \dots, A_{-1}$ are of finite rank. In complex analysis, the coefficient of z^{-1} is the residue at λ. What is the significance here?

Proposition 1.3. *Let $N_\lambda = \bigcup_{k=1}^{\infty} \ker(T - \lambda I)^k$ be the generalized eigenspace of T associated to λ. Then $-A_{-1}$ is a projection onto N_λ. So N_λ is finite dimensional.*

We will prove this next time.